BOUNDARY CONDITIONS INFLUENCE ON COMPRESSIVE STIFFNESS OF ELASTOMERIC ISOLATORS

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Abstract. Elastomers (rubber and rubber-like materials) are widely used in machine building, shipbuilding, civil engineering due to their specific properties: high elasticity, resistance to environmental factors, good dynamic behaviour. Rubberware find an application as compensation devices, isolators, bumpers, shock absorbers, dampers and so on. The important parameters of rubber devices are compression, bending and shear stiffness – the dependence between imposed force and received deformation. In this paper rubber blocks of cylindrical forms under static axial compressive load are considered. Rubber cylinders are considered of finite length with a stress-free side surface and the different types of boundary conditions at the ends, that is, with different methods of fixing the end portions: fixed or bonding by gluing, free support with friction, fixed in the cap. For each case the theoretical dependence of the axial displacement on acting forces (compressive stiffness) are derived using the principle of minimum total potential energy with help of Ritz's method. Five samples of cylindrical rubber blocks with different end conditions were manufactured and tested on loading machine HB Zwick Roell Z-150. The results of analytical solutions are compared with experimental data.

Keywords: elastomers, rubber block, compression stiffness, variational principle, bumper, isolator.

Introduction

Natural and synthetic rubber and rubber-like materials (elastomers) exhibit specific properties: high elasticity, low volume compressibility (Poisson's ratio is 0.48÷0.50), linear relationship between stress and strain in the domain of small strain (up to of 15÷20 %), resistance to environmental factors, good dynamic behaviour. Elastomeric materials find an application in engineering as isolators, dampers, shock absorbers compensation devices, cushions and so on [1-6]. Rubber blocks, consisting of rubber elements and two supporting rigid plates, are widely used in many engineering applications. They have various shapes, various mounting technique and different loading directions, depending on the application.

In this paper the axisymmetrical rubber blocks, made of rubber cylinder placed between two undeformable plates, under axial compressive loading with different methods of supporting the end portions are considered. Different methods of supporting impose different boundary condition for calculation and design the rubber blocks. In this study six variants of boundary conditions are discussed: 1) with two fixed (rigid bonded) plates; 2) with one fixed plate and one friction contact plate, 3) with two fixed rigid caps; 4) with one fixed cap and one fixed plate; 5) with one fixed cap and one friction contact plate; 6) with two friction contact plates. Schematic diagrams of rubber block fastening cases study are given in Fig. 1.

Calculation of the stiffness of rubber elements is a necessary stage of such devices design. Research of the behavior of rubber elements began more than fifty years ago, many researchers study the behaviour of bonded elastic layers [7-13]. There is many publications about the rubber block between fixed-fixed contact simulation, the problem of the other kind of contact have been a little investigated, in spite of the fact of their wide application.

Gent et al. [7-9] developed the approach to stiffness definition named "pressure method". It is accepted that the total displacement of a bonded rubber layer subject to uniform compression is composed of the superposition of two simple displacements: pure homogeneous compression of the corresponding unbonded rubber layer and additional shear displacements to keep the bonded surfaces in their original positions. Gent's "pressure method" is widely used approach since his formulation usually leads to relatively simple expressions, it gives good results for thin rubber layer.

The asymptotic theory of elastomeric layers proposed by Malkov V.M. [14] in many cases allows to receive only numerical solution because of closed form expressions deriving is very complicated. Since precise analytical methods for solving often pose certain problems, an approximate analytical and numerical methods, based on variational principles, are of considerable interest [6; 15-17].

In this study to determine the compressive stiffness of the rubber blocks variational Ritz method are applied, using a principle of minimum of total potential energy [15-17]. To determine the stiffness at small deformations the linear theory of elasticity is used [15].

The objective of this work is to estimate compressive stiffness, i.e. theoretical dependence of the displacement on the acting forces, for the rubber blocks with boundary conditions shown in Fig. 1.

Materials and methods

For compressive stiffness estimation for each case study of boundary conditions Ritz method are applied, using a minimization of the total potential energy functional. As the required functions the displacement functions are selected; for axially symmetric problems cylindrical coordinates system are chosen and the sought –for functions of displacement (*u, v, w*) are set as the functions of the coordinates (*r, θ, z*) and the constants to be determined (C_k). Coordinate system and scheme of rubber block displacement are shown in Fig. 2.

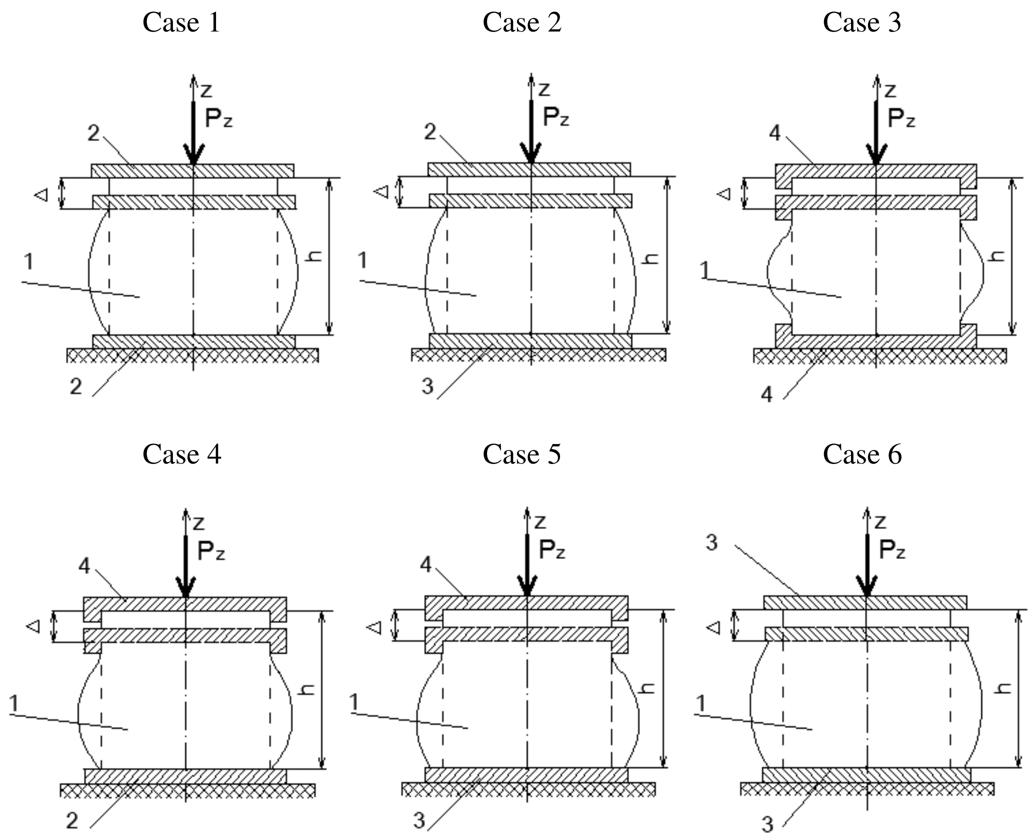

Fig. 1. **Schematic diagrams of rubber block boundary conditions cases** 1 – rubber block; 2 – fixed plate; 3 – friction contact plate; 4 – fixed cap; *h* – rubber block initial height; Δ – axial displacement

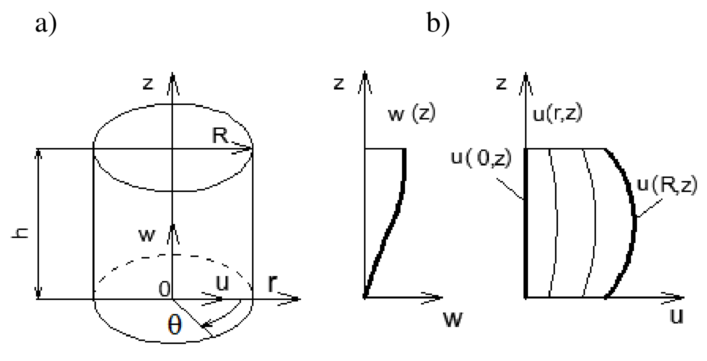

Fig. 2. **Cylindrical coordinate system of rubber block (a), displacement functions for case 1 (b)**

Displacement functions must satisfy the main boundary conditions, as well as the symmetry and the presence of extremum must be taken into account; in our case for simplicity we choose:

$$u = C_1 r f_1(z),\ v = 0,\ w = C_2 f_2(z) \text{ and } f_1 = \frac{\partial f_2}{\partial z}.$$

Stiffness interpreted through force – displacement dependence Δ = (*P*) is found using the principle of a minimum of total potential energy Π:

$$\Pi = U - P\Delta,$$

where *U* – potential energy of deformation of elastomer;
Δ – displacement of elastomeric element.
P – axisimmetrical loading

$$\Pi = G\int_V \left[\left(\varepsilon_r^{\ 2} + \varepsilon_\theta^{\ 2} + \varepsilon_z^{\ 2}\right) + 2\varepsilon_{rz}^{\ 2} + s\left(\varepsilon_r + \varepsilon_\theta + \varepsilon_z\right)\right] dV - P\Delta, \quad (1)$$

$$\varepsilon_r = \frac{\partial u}{\partial r},\ \varepsilon_\theta = \frac{u}{r},\ \varepsilon_z = \frac{\partial w}{\partial z},\ \varepsilon_{rz} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right),\ s = -\frac{P}{\pi R^2 G}, \quad (2)$$

where *G* – variable one, units;
u, w – displacement functions along axis of cylindrical coordinate system *r, z*;
s – function of relative hydrostatic pressure;
V – volumes of rubber.

For incompressible material (Poisson's ratio $\mu = 0.5$):

$$\varepsilon_r + \varepsilon_\theta + \varepsilon_z = 0. \quad (3)$$

Taking into account weak compressibility of elastomeric material ($\mu \neq 0.5$) we can write:

$$\int_V \left(\varepsilon_r + \varepsilon_\theta + \varepsilon_z\right) dV = \frac{3(1-2\mu)}{4(1+\mu)}\int_V s\,dV. \quad (4)$$

Constants C_k as functions of Δ are determined from the boundary conditions and conditions (3) or (4), then are substituted to equations (2). The potential energy equation (1) take a form:

$$\Pi = G\int_V \left[\left(\varepsilon_r^{\ 2} + \varepsilon_\theta^{\ 2} + \varepsilon_z^{\ 2}\right) + 2\varepsilon_{rz}^{\ 2}\right] r\,dr\,d\theta\,dz - P\Delta. \quad (5)$$

After volume integration of equation (5) functional Π becomes a function of displacement Δ, then Δ is determined from the condition of the function *Π* minimum:

$$\frac{d\Pi(\Delta)}{d\Delta} = 0.$$

The results of dependence Δ = *f* (*P*) definition for all presented in Fig. 1cases are given below.

Case study 1: fixed plate – fixed plate

Geometrical boundary condition:

$$u(r, z=0) = u(r, z=h) = 0,\ w(r, z=0) = 0,\ w(r, z=h) = -\Delta,$$

$$\frac{\partial u}{\partial z}(z=0) \neq 0,\ \frac{\partial u}{\partial z}(z=h) \neq 0.$$

Displacement functions:

$$u(r,z) = C_1 r\left(z^2 - zh\right),\ v(r,z) = 0,\ w(r,z) = C_2\left(\frac{z^3}{3} - \frac{z^2 h}{2}\right).$$

Constants of displacement functions and $\Delta - P$ dependence for uncompressible material ($\mu = 0.5$):

$$C_1 = -\frac{3\Delta}{h^3},\ C_2 = \frac{6\Delta}{h^3},\ \Delta_1 = \frac{Ph}{\pi R^2 3G}\frac{1}{\left(1.2+0.5\rho^2\right)},\ \text{where } \rho = \frac{R}{h}. \tag{6}$$

Constants C_1, C_2 and $\Delta - P$ dependence in the case of compressibility accounting ($\mu \neq 0.5$):

$$C_{1\mu} = \frac{9(1-2\mu)}{4(1+\mu)}\frac{P}{\pi R^2 G h^2} - \frac{3\Delta}{h^3},\ C_{2\mu} = \frac{6\Delta}{h^3},\ \Delta_{1\mu} = \frac{Ph}{\pi R^2 3G}\frac{1+(1-2\mu)3\left(1.2+0.5\rho^2\right)}{\left(1.2+0.5\rho^2\right)}. \tag{7}$$

For all next cases formula for $\Delta - P$ dependence for in the case of $\mu \neq 0.5$ is received by the same way:

$$\text{if } \Delta_i = \frac{Ph}{\pi R^2 3G}\frac{1}{g_i(\rho)},\ \text{then } \Delta_{i\mu} = \frac{Ph}{\pi R^2 3G}\frac{1+(1+2\mu)3g_i(\rho)}{g_i(\rho)}, \tag{8}$$

where $g_i(\rho)$ – compressive stiffness increasing factor.

Case study 2: upper fixed plate – friction contact plate lower

Geometrical boundary conditions:

$$u(r,0) \neq 0,\ u(r,h) = 0,\ w(0) = 0,\ w(h) = -\Delta.$$

Displacement functions:

$$u(r,z) = C_1 r\left(z^2 - h^2\right),\ v(r,z) = 0,\ w(z) = C_2\left(\frac{z^3}{3} - zh^2\right).$$

Displacement functions constants and $\Delta - P$ dependence for incompressible material:

$$C_1 = -\frac{3}{4}\frac{\Delta}{h^3},\ C_2 = \frac{3}{2}\frac{\Delta}{h^3},\ \Delta_2 = \frac{Ph}{\pi R^2 3G}\frac{1}{\left(1.2+0.125\rho^2\right)}. \tag{9}$$

Case study 3: fixed cap – fixed cap

Geometrical boundary conditions:

$$u(r,z=0) = u(r,z=h) = 0,\ \frac{\partial u}{\partial r}(z=0) = 0,\ \frac{\partial u}{\partial r}(z=h) = 0,$$

$$w(r,z=0) = 0,\ w(r,z=h) = -\Delta.$$

Displacement functions:

$$u(r,z) = C_1 r\left(z^4 - 2z^3 h + z^2 h^2\right),\ v(r,z) = 0,\ w(r,z) = C_2\left(\frac{z^5}{5} - \frac{z^4 h}{2} + \frac{z^2 h^2}{3}\right).$$

Displacement functions constants and $\Delta - P$ dependence for incompressible material:

$$C_1 = 15\frac{\Delta}{h^5},\ C_2 = -30\frac{\Delta}{h^5},\ \Delta_3 = \frac{Ph}{\pi R^2 3G}\frac{1}{\left(1.429+0.714\rho^2\right)}. \tag{10}$$

Case study 4: upper fixed cap – fixed plate lower

Geometrical boundary conditions:

$$u(r,z=0) = u(r,z=h) = 0,\ \frac{\partial u}{\partial r}(z=0) = 0,\ w(r,z=0) = 0,\ w(r,z=h) = -\Delta.$$

Displacement functions:

$$u(r,z) = C_1 r\left(z^3 - z^2 h\right),\ v(r,z) = 0,\ w(r,z) = C_2\left(3z^2 - 2zh\right).$$

Displacement functions constants and *Δ* - *P* dependence for incompressible material:

$$C_1 = -6\frac{\Delta}{h^4},\ C_2 = 12\frac{\Delta}{h^4},\ \Delta_4 = \frac{Ph}{\pi R^2 3G}\frac{1}{\left(1.371+0.80\rho^2\right)}. \tag{11}$$

Case study 5: upper fixed cap – friction contact plate lower

Geometrical boundary conditions:

$$u(r,z=0)\neq 0,\ u(r,z=h)=0,\ w(r,z=0)=0,\ w(r,z=h)=-\Delta .$$

Displacement functions:

$$u(r,z) = C_1 r\left(2z^3 - 3z^2 h + h^3\right),\ v(r,z)=0,\ w(z) = C_2\left(\frac{1}{2}z^4 - z^3 h + zh^3\right).$$

Displacement functions constants and *Δ* - *P* dependence for incompressible material:

$$C_1 = \frac{\Delta}{h^4},\ C_2 = 2\frac{\Delta}{h^4},\ \Delta_5 = \frac{Ph}{\pi R^2 3G}\frac{1}{\left(1.468+0.20\rho^2\right)}. \tag{12}$$

Case study 6: friction – friction contact plate

Geometrical boundary conditions:

$$u(r,z=0)\neq 0,\ u(r,z=h)\neq 0,\ w(r,z=0)=0,\ w(r,z=h)=-\Delta .$$

Displacement functions:

$$u(r,z) = C_1 r\left(z^2 - zh + h^2\right),\ v(r,z)=0,\ w(z) = C_2\left(\frac{z^3}{3} - \frac{z^2 h}{2} + zh^2\right).$$

Displacement functions constants and *Δ* - *P* dependence for incompressible material:

$$C_1 = -\frac{3}{5}\frac{\Delta}{h^3},\ C_2 = -\frac{6}{5}\frac{\Delta}{h^3},\ \Delta_6 = \frac{Ph}{\pi R^2 3G}\frac{1}{\left(1.008+0.02\rho^2\right)}. \tag{13}$$

As far as $3G \cong E$ (*E* – Young's modulus) then *Δ* - *P* dependence may be written as:

$$\Delta_i = \frac{Ph}{\pi R^2 3G}\frac{1}{g_i(\rho)} = \frac{Ph}{\pi R^2 E}\frac{1}{g_i(\rho)} = \frac{Ph}{\pi R^2 E_a},$$

where E_a – may be called as apparent Young's modulus.

Gent and Lindley [8] and Gent and Meinecke [9] showed that an apparent modulus of elasticity E_a under compressive loadings is a function of Young's modulus *E* and shape factor *s,* defined as the ratio of one loaded area to the total stress free area. To obtain a closed form solution Gent and others introduced next assumptions: rubber is a linear elastic incompressible material; vertical normal stress is constant along the vertical direction; the deformed shape of free side become parabolic; planes parallel to the end plates remained plane before and after deformations. Total displacement is superposition of two simple deformations: pure homogeneous and lateral due to a hydrostatic pressure.

$$E_a = E(1+2s^2),$$

where *s* – shape factor ($s = R/2h$), using agreed in this paper notation $s = 0.5R/h = 0.5\rho$

$$\Delta_1 = \frac{Ph}{\pi R^2 E\left(1+2s^2\right)} = \frac{Ph}{\pi R^2 E\left(1+0.5\rho^2\right)}. \tag{14}$$

Horton, Tupholme and Gover [18] derived formula for apparent modulus of elasticity for the axial deformation of axi-symmetric discs, which is coincided with presented in our work:

$$E_a = E(1.2+2s^2).$$

For fixed-fixed boundary conditions Malkov V.M. proposed to use the asymptotic methods for elastomeric materials based on linear theory of elasticity, which allows in many cases to receive only numerical solution because of closed form expressions deriving is very complicated. [14]. The exact solutions for cylindrical rubber layer compression stiffness, received by Ormonbekov T.O. et al. [19]:

$$C_{com} = \frac{\pi R^2 K}{h}\left[1 - \frac{2I_1(\lambda)}{\lambda I_0(\lambda)}\right], \text{ where } \lambda = \sqrt{12C}\ ,\ C = \frac{G}{K\varepsilon^2},\ \varepsilon = \frac{h}{R}, \tag{15}$$

where K – bulk modulus of rubber,
I_0, I_1 – modified Bessel function of the first kind of zeroth and first order.

Identical equation was presented by Kelly J.M. for the compressive stiffness definition, derived by incorporating the material compressibility directly into the "pressure method" [20].

Results and discussion

1. Plots of dependence of compressive stiffness increasing factor on the ratio *R/h* is given in Fig. 3 for two variants: a) for incompressible material (μ = 0.5), b) for material with μ = 0.493 taking into account weak compressibility. As it is seen from the diagrams if *R/h* less than 3 influence of Poisson's ratio is negligible.

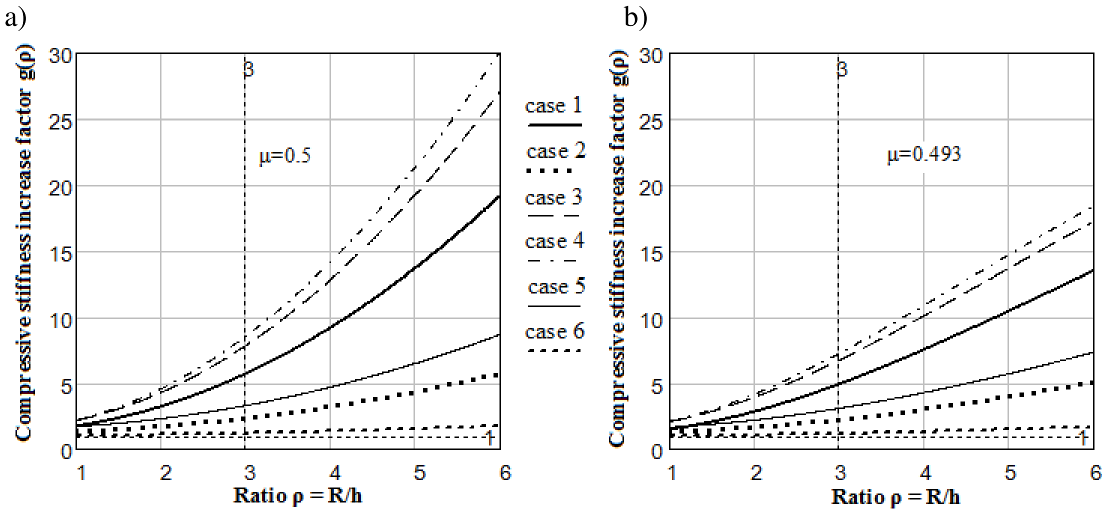

Fig. 3. **Plots of dependence of compressive stiffness increasing factor on *R/h* ratio:** a – without accounting weak compressibility, b – taking into account weak compressibility

2. Compressive tests of five samples with boundary conditions case 1÷5 was carried out on loading machine HB Zwick/Roell Z-150 in Riga Technical University laboratory. Rubber blocks of cylindrical form with R = 36 mm, h = 12 mm, were produced from rubber with G = 0.52 MPa and μ = 0.493. Fixed contacts were implemented by adhesive bonded to metal plate. Results of experiments and theoretical calculations are given in Table 1.

Table 1

Results of experiments and theoretical calculations

Load, N	Displacement	Compressive strain, mm				
		Sample 1	Sample 2	Sample 3	Sample 4	Sample 5
2000	Δ_{exp}	0.698	1.705	0.483	0.472	1.268
	$\Delta_t\ \mu = 0.5$	0.663	1.626	0.481	0.449	1.151
	$\Delta_t \mu = 0.493$	0.716	1.678	0.534	0.494	1.203
4000	Δ_{exp}	1.371	3.613	0.978	0.952	2.524
	$\Delta_t\ \mu = 0.5$	1.326	3.252	0.962	0.898	2.302
	$\Delta_t\ \mu = 0.493$	1.432	3.357	1.068	0.988	2.406

Experimental results approve the theoretical calculation. Displacements calculated in accordance with Gent's formula (14): Δ = 0.687mm for P = 2 kN and Δ = 1.374 mm for P = 4 kN, which give very good agreement with test results. Displacements calculated in accordance with formula (15): $\Delta = 0.91$mm for $P = 2$ kN and $\Delta = 1.821$mm for $P = 4$ kN. Difference appears because this method aimed to thin elastomeric layer ($\rho > 5$).

Conclusions

1. In given work one of the variotional method of compressive stiffness definition for cylindrical rubber element with different boundary condition is presented. Solution was based on the principle of minimum total potential energy of deformation, using Ritz method. Small deformations was considered without taking into account weak compressibility of the rubber (Poisson's ratio was assumed as 0.5) and taking it into account. It is shown that increasing the ratio $R/h = \rho$ leads to compressive stiffness increase. The thinner layer of elastomer, the greater the effect of Poisson's ratio. If the ratio $R/h > 3 \div 5$ it is necessary to take into account rubber weak compressibility.
2. The results of analytical solutions are compared with experimental data, theoretical results have good agreement with experimental data. This proved the applicabillity of Ritz's method to solving such problems.

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